

# THE MARTINGALE PROBLEM FOR MARKOV SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

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**ABSTRACT.** Under suitable assumptions of regularity and non-degeneracy on the covariance of the driving additive noise, any Markov solution to the stochastic Navier-Stokes equations has an associated generator of the diffusion and is the unique solution to the corresponding martingale problem. Some elementary examples are discussed to interpret these results.

## 1. INTRODUCTION

For some interesting stochastic partial differential equations, such as the three dimensional Navier-Stokes equations, well-posedness of the associated martingale problem is still an open problem. If on one hand this corresponds to a poor understanding of the deterministic dynamics (see for instance Feffermann [7]), on the other hand there is still the possibility that the stochastic problem may have better uniqueness properties as in the finite dimensional case, see for example [23, Chapter 8], see also [8] for a review in comparison with the infinite dimension and [9] for a infinite dimensional positive result.

With these uniqueness problems in mind, it is then reasonable, and sometimes useful, to consider special solutions with additional properties. We focus here on solutions to the stochastic Navier-Stokes equations,

$$(1.1) \quad \begin{cases} \dot{u} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = \eta, \\ \operatorname{div} u = 0, \end{cases}$$

which are Markov processes, and we give a short account of the theory introduced in [11] and [13], and developed in [12], [21] (see also [1], where such ideas have been applied on a stochastic fourth order parabolic equation driven by space-time white noise and modelling surface growth). It turns out that, under suitable assumptions of regularity and non-degeneracy on the covariance of the noise, all Markov solutions to (1.1) are strong Feller and converge, exponentially fast, to a unique invariant measure. Similar (and sometimes better) results have been already obtained by Da Prato & Debussche [3] (see also [5], [20] and [4]) with a completely different method.

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The aim of this paper is to understand the martingale problem associated to any of the Markov processes which are solutions to (1.1). We prove that each of these Markov processes has a generator and it is the unique solution of the martingale problem associated to the generator of its own dynamics. It remains completely open to understand what these generators have in common with the formal generator

$$\mathcal{L}^* = \frac{1}{2} \text{Tr}[QD^2] - \langle -v\nabla u + (u \cdot \nabla), D \rangle.$$

where  $Q$  is the spatial covariance of the noise, and in Section 3.2 we make an attempt to connect  $\mathcal{L}^*$  with the generators.

In order to compare all such results, in the final part of the paper we give a few instructive examples (essentially taken from Stroock & Yor [24]) of one dimensional stochastic differential equations where uniqueness is not ensured. All these examples show that it is possible to have problems where different solutions are Feller (and each of them has a unique invariant measure, see Section 4.2) or even strong Feller (Section 4.3).

In conclusion, regardless of the improvement gained with the Markov approach ([3] and [13]), the problem remains completely open. The examples presented here strengthen the belief that we must understand the behaviour of these solutions when they approach *blow-up* epochs (see for example [10] for an attempt in this direction).

The paper is organised as follows. Section 2 contains a short summary of results on Markov solutions for the Navier-Stokes equations. Existence of the generator and all details on the martingale problem can be found in Section 3. Finally, the examples are presented in the last section.

## 2. MARKOV SOLUTIONS FOR THE NAVIER-STOKES EQUATIONS

In this section we give a short account of the ideas introduced in [11], [12], [13], [21]. We focus on the equations on the three-dimensional torus  $\mathbb{T}_3 = [0, 2\pi]^3$  with periodic boundary conditions.

We fix some notations we shall use throughout the paper and we refer to Temam [25] for a detailed account of all the definitions. Let  $\mathcal{D}^\infty$  be the space of infinitely differentiable divergence-free periodic vector fields  $\phi : \mathbf{R}^3 \rightarrow \mathbf{R}^3$  with mean zero on  $\mathbb{T}_3$ . Denote by  $H$  the closure of  $\mathcal{D}^\infty$  in  $L^2(\mathbb{T}_3, \mathbf{R}^3)$  and by  $V$  the closure in  $H^1(\mathbb{T}_3, \mathbf{R}^3)$ . Denote by  $A$ , with domain  $D(A)$ , the *Stokes* operator and define the bi-linear operator  $B : V \times V \rightarrow V'$  as the projection onto  $H$  of the nonlinearity of equation (1.1). Let  $Q$  be a linear bounded symmetric positive operator on  $H$  with finite trace. Consider finally the abstract form of problem 1.1,

$$(2.1) \quad du + vAu + B(u, u) = Q^{\frac{1}{2}}dW,$$

where  $W$  is a cylindrical Wiener process on  $H$ .

The first step is to define a solution to the above equation (2.1). To this end, let  $\Omega = C([0, \infty); D(A)')$ , let  $\mathcal{B}$  be the Borel  $\sigma$ -field on  $\Omega$  and let  $\xi : \Omega \rightarrow D(A)'$  be the canonical process on  $\Omega$  (that is,  $\xi_t(\omega) = \omega(t)$ ). A filtration can be defined on  $\mathcal{B}$  as  $\mathcal{B}_t = \sigma(\xi_s : 0 \leq s \leq t)$ .

For every  $\varphi \in \mathcal{D}^\infty$  consider the process  $(M_t^\varphi)_{t \geq 0}$  on  $\Omega$  defined for  $t \geq 0$  as

$$(2.2) \quad M_t^\varphi = \langle \xi_t - \xi_0, \varphi \rangle + \nu \int_0^t \langle \xi_s, A\varphi \rangle ds - \int_0^t \langle B(\xi_s, \varphi), \xi_s \rangle ds.$$

Moreover, for every  $n \in \mathbf{N}$ ,  $n \geq 1$ , define for  $t \geq 0$  the process

$$(2.3) \quad \mathcal{E}_t^n = |\xi_t|_H^{2n} + 2n\nu \int_0^t |\xi_s|_H^{2n-2} |\xi_s|_V^2 ds - n(2n-1) \text{Tr}[Q] \int_0^t |\xi_s|_H^{2n-2} ds,$$

when  $\xi \in L_{\text{loc}}^\infty([0, \infty); H) \cap L_{\text{loc}}^2([0, \infty); V)$ , and  $\infty$  elsewhere.

**Definition 2.1.** Given  $\mu_0 \in \text{Pr}(H)$ , a probability  $P$  on  $(\Omega, \mathcal{B})$  is a solution starting at  $\mu_0$  to the martingale problem associated to the Navier-Stokes equations (2.1) if

- $P[L_{\text{loc}}^\infty([0, \infty); H) \cap L_{\text{loc}}^2([0, \infty); V)] = 1$ ,
- for each  $\varphi \in \mathcal{D}^\infty$  the process  $M_t^\varphi$  is square integrable and  $(M_t^\varphi, \mathcal{B}_t, P)$  is a continuous martingale with quadratic variation  $[M^\varphi]_t = t |Q^{\frac{1}{2}} \varphi|_H^2$ ,
- for each  $n \geq 1$ , the process  $\mathcal{E}_t^n$  is  $P$ -integrable and for almost every  $s \geq 0$  (including  $s = 0$ ) and all  $t \geq s$ ,

$$\mathbb{E}[\mathcal{E}_t^n | \mathcal{B}_s] \leq \mathcal{E}_s^n,$$

- the marginal of  $P$  at time  $t = 0$  is  $\mu_0$ .

*Remark 2.2.* Goldys, Rockner & Zhang [15] have pointed out that, due to a lack of measurability of conditional probabilities, the condition on the process  $\mathcal{E}$  should be replaced by an analogous condition on  $\sup_{s \leq t} \mathcal{E}_s$ .

The idea behind the existence of Markov solutions is by Krylov [18] (see also Chapter 12 of Stroock & Varadhan [23]). Define for every  $x \in H$  the set  $\mathcal{C}(x)$  of all solutions (according to Definition 2.1 above) starting at  $\delta_x$ .

**Theorem 2.3** (Theorem 4.1, [13]). *There exists a family  $(P_x)_{x \in H}$  of weak martingale solutions such that  $P_x \in \mathcal{C}(x)$  for every  $x \in H$  and the almost sure Markov property holds. More precisely, for every  $x \in H$ , for almost every  $s \geq 0$  (including  $s = 0$ ), for all  $t \geq s$  and all bounded measurable  $\phi : H \rightarrow \mathbf{R}$ ,*

$$(2.4) \quad \mathbb{E}^{P_x}[\phi(\xi'_t) | \mathcal{B}_s] = \mathbb{E}^{P_{\xi_s}}[\phi(\xi'_{t-s})].$$

The set of times where the Markov property fails to hold at some point  $x$  will be called the set of *exceptional times* of  $x$ .

A very short outline of the proof (a complete version is given in [13]) is the following:

- the sets  $(\mathcal{C}(x))_{x \in H}$  satisfy a *set-valued* version of the Markov property,
- given  $\lambda > 0$  and  $f \in C_b(H)$ , the set of maxima of the function

$$P \mapsto \mathcal{R}_{\lambda, f}(P) := \mathbb{E}^P \left[ \int_0^\infty e^{-\lambda t} f(\xi_t) dt \right]$$

satisfies again the *set-valued* version of the Markov property,

- the proof is completed by iterating the above argument over a countable dense set of  $\lambda \in (0, \infty)$  and  $f \in C_b(H)$ .

In particular an arbitrary solution  $P \in \mathcal{C}(x)$  (for some  $x \in H$ ) may not be an element of a Markov solution (for a counterexample, see Proposition 4.2). Moreover, an arbitrary Markov solution may not be *extremal*, that is, may not be obtained with the procedure outlined above (for a counterexample see Proposition 4.3).

So far, the previous theorem ensures the existence of Markov solutions such that the map  $x \mapsto P_x$  is measurable (with respect to the appropriate  $\sigma$ -fields). In order to improve the dependence with respect to the initial condition, the assumptions on the covariance of the noise need to be strengthened. Assume that for some  $\alpha_0 > \frac{1}{6}$  the operator  $A^{\frac{3}{4}+\alpha_0} Q^{\frac{1}{2}}$  is bounded invertible on  $H$ , with bounded inverse. The additional regularity of noise trajectories allows to exploit the following fact: for regular initial conditions there is a unique strong<sup>1</sup> solution up to a random time in which the derivatives of  $u$  blow up. The random time can be approximated monotonically by a sequence of stopping times

$$\tau_x^{(R)} = \inf\{t > 0 : \|u_x^{(R)}(t)\|_{\mathcal{W}_{\alpha_0}}^2 \geq R\}$$

(see below for the definition of  $\mathcal{W}_{\alpha_0}$ ), where  $u_x^{(R)}$  solves

$$(2.5) \quad du_x^{(R)} + (vAu_x^{(R)} + \chi(\frac{1}{R}\|u_x^{(R)}\|_{\mathcal{W}_{\alpha_0}}^2)B(u_x^{(R)}, u_x^{(R)})) dt = Q^{\frac{1}{2}} dW,$$

with initial condition  $x$  and  $\chi$  is a *cut-off* function such that  $\chi(r) \equiv 1$  for  $r \leq 1$  and  $\chi(r) \equiv 0$  for  $r \geq 2$ .

Consider a Markov solution  $(P_x)_{x \in H}$  and define for every  $t \geq 0$  the transition semi-group  $\mathcal{P}_t : \mathcal{B}_b(H) \rightarrow \mathcal{B}_b(H)$  as

$$\mathcal{P}_t \phi(x) = \mathbb{E}^{P_x}[\phi(\xi_t)].$$

**Theorem 2.4** (Theorem 5.11, [13]). *Given a Markov solution  $(P_x)_{x \in H}$ , let  $(\mathcal{P}_t)_{t \geq 0}$  be its transition semigroup. Then for every  $t > 0$  and  $\phi \in \mathcal{B}_b(H)$ ,  $\mathcal{P}_t \phi \in C_b(\mathcal{W}_{\alpha_0})$ .*

The continuity in the above theorem is in the topology of  $\mathcal{W}_{\alpha_0} = D(A^\theta)$ , where  $\theta = \frac{1}{2}(\alpha_0 + 1)$  if  $\alpha_0 < \frac{1}{2}$  and  $\theta = \alpha_0 + \frac{1}{4}$  if  $\alpha_0 \geq \frac{1}{2}$ .

*Remark 2.5* (On regularity). Indeed, it is possible to improve the regularity result given above by relaxing the topology. Under the same assumptions on the noise, one can replace in Theorem 2.4  $\mathcal{W}_{\alpha_0}$  with  $D(A^{1/4+\varepsilon})$  (with arbitrary  $\varepsilon > 0$ ), by using parabolic regularisation. The main idea is that  $D(A^{\frac{1}{4}})$  is a critical space, that is to say, it is the smallest space, in the hierarchy of hilbertian Sobolev spaces, where it is possible to prove existence and uniqueness of local strong solutions for the deterministic version of the problem. This extension is part of a work which is currently in progress.

*Remark 2.6* (On non-degeneracy). The non-degeneracy assumption on the covariance  $Q$  can be slightly relaxed. Assume for instance that  $Q$  is diagonal with respect to the Fourier basis and assume that the range of  $Q$  spans all but a finite number of Fourier modes. It is possible then to prove that any Markov solution is strong Feller (this is a

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<sup>1</sup>The *strong* has to be understood in the PDE sense

work in progress in collaboration with L. Xu). Unfortunately, at least with the method presented here where the strong Feller property is crucial, it does not seem to be possible to consider a noise highly degenerate as, for example, in Hairer & Mattingly [16].

The above regularity result allows to analyse the long time behaviour of any arbitrary Markov solution. The following theorem collects Corollary 3.2 and Theorem 3.3 from [21].

**Theorem 2.7.** *Under the above assumptions on the covariance, every Markov solution  $(P_x)_{x \in H}$  to (1.1) has a unique invariant measure  $\mu$ . Moreover, there are  $c > 0$  and  $a > 0$  such that*

$$\|\mathcal{P}_t^* \delta_{x_0} - \mu\|_{\text{TV}} \leq c e^{-at},$$

for every  $x_0 \in H$ , where  $\|\cdot\|_{\text{TV}}$  is the total variation norm.

It is worth remarking that the above theorem states that uniqueness of the invariant measure holds among all measures which are invariant with respect to the given Markov solution. In different words, different Markov solutions have different invariant measures. The following result, which gathers Corollary 3.5 and Theorem 4.6 of [21], allows to compare invariant measures for different Markov solutions.

**Theorem 2.8.** *All invariant measures associated to Markov solutions are mutually equivalent. Moreover, if all Markov solutions share the same invariant measure, then the martingale problem is well-posed.*

### 3. THE MARTINGALE PROBLEM FOR THE NAVIER-STOKES EQUATIONS

In this section we follow closely Appendix B of Cerrai [2]. Let  $(P_x)_{x \in H}$  be a Markov solution and let  $(\mathcal{P}_t)_{t \geq 0}$  be the associated transition semigroup. In the rest of the section we will denote by  $\mathcal{W}$  the space where the Markov solution is continuous, without stating any dependence on the parameter  $\alpha_0$ . In view of Remark 2.5, there is no loss of generality in doing so.

**Lemma 3.1.** *Let  $\varphi \in \mathcal{B}_b(H)$ , then*

- if  $x \in \mathcal{W}$ , then the map  $t \mapsto \mathcal{P}_t \varphi(x)$  is continuous for all  $t \in [0, \infty)$ ,
- if  $x \in H$ , then the map  $t \mapsto \mathcal{P}_t \varphi(x)$  is continuous for all  $t \in (0, \infty)$ .

*Proof.* If  $x \in \mathcal{W}$ , the statement follows from Lemma 6.6 of [13]. If  $x \in H$  and  $t_0 > 0$ , choose  $\delta > 0$  such that  $t_0 - \delta > 0$  and  $t_0 - \delta$  is not an exceptional time for  $x$ , then

$$\mathcal{P}_t \varphi(x) = \mathbb{E}^{P_x}[(\mathcal{P}_{t-\delta} \varphi)(\xi_\delta)].$$

Since by Lemma 3.7 of [21]  $P_x[\xi_\delta \in \mathcal{W}] = 1$ , by the first statement of the lemma it follows that  $(\mathcal{P}_{t-\delta} \varphi)(\xi_\delta) \rightarrow (\mathcal{P}_{t_0-\delta} \varphi)(\xi_\delta)$   $P_x$ -a. s.. The conclusion follows from Lebesgue theorem.  $\square$

Consider now  $\lambda > 0$  and define the operator  $\mathcal{R}_\lambda : \mathcal{B}_b(H) \rightarrow \mathcal{B}_b(H)$  as

$$\mathcal{R}_\lambda \varphi(x) = \int_0^\infty e^{-\lambda t} \mathcal{P}_t \varphi(x) dt.$$

**Lemma 3.2.** *For every  $\lambda > 0$ ,  $\mathcal{R}_\lambda$  is a bounded operator on  $C_b(\mathcal{W})$ . Moreover, the resolvent identity holds. For every  $\lambda_1, \lambda_2$ ,*

$$\mathcal{R}_{\lambda_1} - \mathcal{R}_{\lambda_2} = (\lambda_2 - \lambda_1) \mathcal{R}_{\lambda_1} \mathcal{R}_{\lambda_2}.$$

*Proof.* Continuity of  $\mathcal{R}_\lambda \phi$  follows from the strong Feller property and Lebesgue theorem (since  $e^{-\lambda t}$  is integrable). Moreover, if  $x \in \mathcal{W}$ ,

$$|\mathcal{R}_\lambda \phi(x)| \leq \int_0^\infty e^{-\lambda t} |\mathcal{P}_t \phi(x)| dt \leq \frac{1}{\lambda} \|\phi\|_\infty.$$

Next, we prove the resolvent identity. Fix  $x \in \mathcal{W}$ , then for a. e.  $s$ ,  $\mathcal{P}_{t+s} \phi(x) = \mathcal{P}_t \mathcal{P}_s \phi(x)$ , and so

$$\begin{aligned} \mathcal{P}_t \mathcal{R}_{\lambda_2} \phi(x) &= \mathbb{E}^{P_x}[\mathcal{R}_{\lambda_2} \phi(\xi_t)] = \mathbb{E}^{P_x}\left[\int_0^\infty e^{-\lambda_2 s} \mathcal{P}_s \phi(\xi_t) ds\right] \\ (3.1) \quad &= \int_0^\infty e^{-\lambda_2 s} \mathbb{E}^{P_x}[\mathcal{P}_s \phi(\xi_t)] ds = \int_0^\infty e^{-\lambda_2 s} \mathcal{P}_{t+s} \phi(x) ds. \end{aligned}$$

In conclusion,

$$\begin{aligned} (\lambda_2 - \lambda_1) \mathcal{R}_{\lambda_1} \mathcal{R}_{\lambda_2} \phi(x) &= (\lambda_2 - \lambda_1) \int_0^\infty e^{-\lambda_1 t} \int_0^\infty e^{-\lambda_2 s} \mathcal{P}_{t+s} \phi(x) ds dt \\ &= (\lambda_2 - \lambda_1) \int_0^\infty e^{-\lambda_1 t} \int_t^\infty e^{-\lambda_2(r-t)} \mathcal{P}_r \phi(x) dr dt \\ &= \int_0^\infty e^{-\lambda_2 r} (e^{(\lambda_2 - \lambda_1)r} - 1) \mathcal{P}_r \phi(x) dr \\ &= \mathcal{R}_{\lambda_1} \phi(x) - \mathcal{R}_{\lambda_2} \phi(x), \end{aligned}$$

and the identity holds.  $\square$

We are finally able to prove existence of the generator.

**Theorem 3.3.** *Given a Markov solution  $(P_x)_{x \in H}$ , there exists a unique closed linear operator  $\mathcal{L} : D(\mathcal{L}) \subset C_b(\mathcal{W}) \rightarrow C_b(\mathcal{W})$  such that for all  $\lambda > 0$  and  $\phi \in C_b(\mathcal{W})$ ,*

$$(3.2) \quad R_\lambda(\mathcal{L}) \phi(x) = \int_0^\infty e^{-\lambda t} \mathcal{P}_t \phi(x) dt,$$

where  $R_\lambda(\mathcal{L})$  is the resolvent of  $\mathcal{L}$ .

*Proof.* By the previous lemma,  $(\mathcal{R}_\lambda)_{\lambda > 0}$  satisfies the resolvent identity. Theorem VIII.4.1 of Yosida [26] ensures then that  $(\mathcal{R}_\lambda)_{\lambda > 0}$  is the resolvent of a linear operator  $\mathcal{L}$  if the kernel  $N(\mathcal{R}_\lambda) = \{0\}$ . In such a case, the domain  $D(\mathcal{L})$  is equal to the range  $R(\mathcal{R}_\lambda)$ , which is independent of  $\lambda$  by the resolvent identity.

We prove that  $N(\mathcal{R}_\lambda) = \{0\}$ . Fix  $\lambda_0 > 0$  and let  $\phi$  be such that  $\mathcal{R}_{\lambda_0} \phi = 0$ . By the resolvent identity it follows that  $\mathcal{R}_\lambda \phi = 0$  for all  $\lambda > 0$ . By inverting the Laplace transform, it follows that  $\mathcal{P}_t \phi(x) = 0$  for all  $x \in \mathcal{W}$  and almost every  $t > 0$  (hence all  $t \geq 0$  by Lemma 3.1). In particular,  $\phi = \mathcal{P}_0 \phi = 0$ .  $\square$

**3.1. The martingale problem.** The computations of the previous section ensure that each Markov solution has a generator. This allows to define the martingale problem associated to this operator.

**Definition 3.4** (Martingale problem). Let  $\mathcal{L}$  be the generator associated to some Markov solution and provided by Theorem 3.3 and let  $x \in \mathcal{W}$ . A probability measure  $P$  on  $(\Omega, \mathcal{B})$  is a solution to the martingale problem associated to  $\mathcal{L}$  and starting at  $x$  if

- $P[\xi_0 = x] = 1$ ,
- for every  $\varphi \in D(\mathcal{L})$ , the process

$$\mathcal{M}_t^\varphi = \varphi(\xi_t) - \int_0^t \mathcal{L}\varphi(\xi_s) ds$$

is a  $P$ -martingale with respect to the natural filtration  $(\mathcal{B}_t)_{t \geq 0}$ .

The aim of this section is to prove that each Markov solution is the unique solution to the martingale problem associated to the corresponding generator. With this aim in mind, we need the following lemma, which is from Appendix B of Cerrai [2]. We give a short account of its proof (which is essentially the same) because the assumptions under which we work are slightly different.

**Lemma 3.5.** For every  $\varphi \in D(\mathcal{L})$  and  $x \in \mathcal{W}$ ,

$$(3.3) \quad \mathcal{P}_t \varphi(x) = \varphi(x) + \int_0^t \mathcal{P}_s \mathcal{L}\varphi(x) ds.$$

In particular,

$$\frac{d}{dt} \mathcal{P}_t \varphi(x) = \mathcal{P}_t \mathcal{L}\varphi(x) = \mathcal{L} \mathcal{P}_t \varphi(x).$$

*Proof.* By formula (3.1), it follows that  $\mathcal{P}_t \mathcal{R}_\lambda = \mathcal{R}_\lambda \mathcal{P}_t$ . Hence,  $\mathcal{P}_t(D(\mathcal{L})) \subset D(\mathcal{L})$  since  $D(\mathcal{L}) = \mathcal{R}_\lambda(C_b(\mathcal{W}))$  and so  $\mathcal{L} \mathcal{P}_t = \mathcal{P}_t \mathcal{L}$ .

We prove (3.3). Let  $\varphi \in D(\mathcal{L})$ ,  $x \in \mathcal{W}$  and  $\lambda > 0$ , then

$$\varphi(x) = \mathcal{R}_\lambda(\lambda I - \mathcal{L})\varphi(x) = \lambda \int_0^\infty e^{-\lambda t} \mathcal{P}_t \varphi(x) dt - \int_0^\infty e^{-\lambda t} \mathcal{P}_t \mathcal{L}\varphi(x) dt$$

and so by Fubini theorem,

$$\begin{aligned} \int_0^\infty e^{-\lambda t} (\mathcal{P}_t \varphi(x) - \varphi(x)) dt &= \int_0^\infty \frac{1}{\lambda} e^{-\lambda t} \mathcal{P}_t \mathcal{L}\varphi(x) dt \\ &= \int_0^\infty \int_t^\infty e^{-\lambda s} \mathcal{P}_t \mathcal{L}\varphi(x) ds dt \\ &= \int_0^\infty e^{-\lambda s} \int_0^\infty \mathcal{P}_t \mathcal{L}\varphi(x) dt ds. \end{aligned}$$

By inverting the Laplace transform and using Lemma 3.1, (3.3) follows.  $\square$

**Theorem 3.6.** Let  $(P_x)_{x \in \mathcal{W}}$  be a Markov solution and let  $\mathcal{L}$  be the associated generator. Then the family  $(P_x)_{x \in \mathcal{W}}$  is the unique solution to the martingale problem associated to  $\mathcal{L}$ .



*Proof.* Both proofs of existence and uniqueness are classical (see for example Stroock & Varadhan [23]), we give a proof for the interested reader. First, we prove that  $(P_x)_{x \in \mathcal{W}}$  is a solution to the martingale problem. The Markov property (2.4) ensures that

$$\mathbb{E}^{P_x}[\varphi(\xi_t) - \mathcal{P}_{t-s}\varphi(\xi_s) | \mathcal{B}_s] = 0,$$

while Lemma 3.5 implies that,  $P_x$ -a. s.,

$$\varphi(\xi_s) = \mathcal{P}_{t-s}\varphi(\xi_s) + \int_s^t \mathcal{P}_{r-s}\mathcal{L}\varphi(\xi_s) dr.$$

Hence,

$$\begin{aligned} \mathbb{E}^{P_x}[\mathcal{M}_t^\varphi - \mathcal{M}_s^\varphi | \mathcal{B}_s] &= \mathbb{E}^{P_x}[\varphi(\xi_t) - \varphi(\xi_s) - \int_s^t \mathcal{L}\varphi(\xi_r) dr | \mathcal{B}_s] \\ &= \mathbb{E}^{P_x}[\varphi(\xi_t) - \mathcal{P}_{t-s}\varphi(\xi_s) - \int_s^t (\mathcal{L}\varphi(\xi_r) - \mathcal{P}_{r-s}\mathcal{L}\varphi(\xi_s)) | \mathcal{B}_s] \\ &= 0. \end{aligned}$$

Next, we prove that  $P_x$  is the unique solution. Let  $P$  be a solution to the martingale problem starting at  $x$ , let  $\phi \in C_b(\mathcal{W})$  and set  $\varphi = R_\lambda(\mathcal{L})\phi$ . By definition of solution,

$$\varphi(x) = \mathbb{E}^P[\varphi(\xi_t) - \int_0^t \mathcal{L}\varphi(\xi_s) ds]$$

and so by multiplying by  $\lambda e^{-\lambda t}$  and integrating by parts,

$$\begin{aligned} \varphi(x) &= \mathbb{E}^P\left[\int_0^\infty e^{-\lambda t} (\lambda I - \mathcal{L})\varphi(\xi_t) dt\right] \\ &= \mathbb{E}^P\left[\int_0^\infty e^{-\lambda t} \phi(\xi_t) dt\right]. \end{aligned}$$

By using (3.2) and inverting the Laplace transform, it follows that  $\mathbb{E}^P[\phi(\xi_t)] = \mathcal{P}_t\phi(x)$ . Since for a Markov process uniqueness of one-dimensional distributions implies uniqueness of laws, the theorem is proved.  $\square$

*Remark 3.7.* Da Prato and Debussche [4] give a stricter definition of solution to the martingale problem, due to the better knowledge they have on their Markov solution, which is obtained via Galerkin approximations (see [3]).

**3.2. What can we say of the generator.** So far, we have proved that any Markov solution  $(P_x)_{x \in H}$  is the unique solution to the martingale problem associated to the generator of the transition semigroup. On the other hand, the formal expression of the generator associated to (2.1) is

$$\mathcal{L}^*\varphi(x) = \frac{1}{2} \text{Tr}[Q D^2\varphi](x) - \langle \mathbf{v}Ax + B(x, x), D\varphi(x) \rangle.$$

In this section we shall try to understand (although without success) if there is any relation between  $\mathcal{L}^*$  and the generator  $\mathcal{L}$  of an arbitrary Markov solution.



To this aim, fix a Markov solution  $(P_x)_{x \in H}$  and let  $\mathcal{L}$  be the associated generator. It is useful to notice that the generator  $\mathcal{L}$  can be characterised (see Da Prato & Debussche [4]) in the following way,

$$D(\mathcal{L}) = \left\{ \varphi \in C_b(\mathcal{W}) : \lim_{\varepsilon \rightarrow 0} \frac{P_\varepsilon \varphi(x) - \varphi(x)}{\varepsilon} \text{ exists } \forall x \in \mathcal{W} \text{ and is in } C_b(\mathcal{W}) \right\}$$

and

$$\mathcal{L}\varphi(x) = \lim_{\varepsilon \rightarrow 0} \frac{P_\varepsilon \varphi(x) - \varphi(x)}{\varepsilon}.$$

Let  $(\mathcal{P}_t^{(R)})_{t \geq 0}$  the Markov semigroup associated to the cut-off problem (2.5) and let  $\mathcal{L}^{(R)}$  be the corresponding generator.

**Lemma 3.8.** *Given  $R \geq 1$ ,*

- *if  $\phi \in D(\mathcal{L}^{(R)})$ , then for every  $|x|_{\mathcal{W}}^2 < R$ ,  $\mathcal{L}^{(R)}\phi(x) = \mathcal{L}^*\phi(x)$ ,*
- *if  $\phi \in D(\mathcal{L}^{(R)})$ , then for every  $|x|_{\mathcal{W}}^2 < R$ ,*

$$\lim_{t \rightarrow 0} \frac{1}{t} (\mathcal{P}_t \phi(x) - \phi(x)) = \mathcal{L}^{(R)}\phi(x),$$

- *if  $\phi \in D(\mathcal{L})$ , then for every  $|x|_{\mathcal{W}}^2 < R$ ,*

$$\lim_{t \rightarrow 0} \frac{1}{t} (\mathcal{P}_t \phi(x) - \phi(x)) = \mathcal{L}\phi(x),$$

*Proof.* The first property is easy. The second and third property follow from

$$\begin{aligned} (3.4) \quad \left| \frac{\mathcal{P}_t^{(R)}\phi(x) - \phi(x)}{t} - \frac{\mathcal{P}_t\phi(x) - \phi(x)}{t} \right| &= \left| \frac{\mathcal{P}_t^{(R)}\phi(x) - \mathcal{P}_t\phi(x)}{t} \right| \\ &\leq \frac{2}{t} \|\phi\|_\infty P_x^{(R)}[\tau_x^{(R)} < t] \\ &\leq \frac{c_1}{t} \|\phi\|_\infty e^{-c_2 \frac{R^2}{t}}, \end{aligned}$$

for  $t \leq cR^{-\gamma}$  (for some  $c > 0$  and  $\gamma > 0$ ), where the first inequality follows from Lemma 5.9 of [13] (see also part 1 of Theorem A.1 in [21]) and the second inequality follows from Proposition 11 of [12] (see also part 2 of Theorem A.1 in [21]).  $\square$

Based on this lemma, the following proposition gives a (almost elementary) condition for the generator  $\mathcal{L}$  to be equal to the formal expression  $\mathcal{L}^*$ .

**Proposition 3.9.** *Let*

$$\mathcal{E} = \left\{ \phi \in D(\mathcal{L}) : \text{there is } R_n \uparrow \infty \text{ s.t. } \phi \in \bigcap_{n \in \mathbb{N}} D(\mathcal{L}^{(R_n)}) \text{ and } \sup_{n \in \mathbb{N}} \|\mathcal{L}^{(R_n)}\phi\|_\infty \right\}.$$

*Then  $\mathcal{L}\phi = \mathcal{L}^*\phi$  for every  $\phi \in \mathcal{E}$ .*

*Proof.* The property follows from inequality (3.4), since  $\frac{1}{t} (\mathcal{P}_t^{(R_n)}\phi(x) - \phi(x))$  is bounded because

$$\left| \frac{\mathcal{P}_t^{(R_n)}\phi(x) - \phi(x)}{t} \right| = \left| \int_0^t \mathcal{P}_s^{(R_n)} \mathcal{L}^{(R_n)}\phi(x) ds \right| \leq \sup_{n \in \mathbb{N}} \|\mathcal{L}^{(R_n)}\phi\|_\infty$$

and  $\frac{1}{t}(\mathcal{P}_t\phi(x) - \phi(x))$  is bounded by the alternative characterisation of  $\mathcal{L}$  given above.  $\square$

*Remark 3.10.* No better conclusion can be drawn with such generality (see Da Prato & Debussche [4] for some related results). The argument missing in this analysis is, essentially, a better estimate of tails of the stopping time  $\tau_x^{(R)}$ , which is used in formula (3.4) to estimate the distance from the generator to the cut-off problem.

#### 4. SOME ELEMENTARY EXAMPLES OF MARKOV SOLUTIONS

In this last section we present some elementary examples from the theory of (deterministic and stochastic) differential equations. We wish to compare these with all results on the stochastic Navier-Stokes equations given in the previous sections.

The first example is a revisitation of a classical example of non-uniqueness in ordinary differential equations, where it is easy to characterise all Markov solutions (compare Proposition 4.3 with Theorem 2.8).

The second example is taken from a paper by Girsanov [14], where all Markov solutions are Feller and it is possible to list the generators of all such solutions (compare with Section 3.2).

The last example has been presented by Stroock & Yor [24] and its main interest is that there are two (strong) Markov solutions which are both strong Feller (compare with Theorem 2.4).

**4.1. An example from elementary calculus.** Consider the following differential equation

$$(4.1) \quad \dot{X} = -X + \sqrt{X},$$

with initial condition  $X(0) = x \in [0, 1]$ . The problem has a unique solution  $X_x(\cdot)$  for  $x \neq 0$  and the family of solutions

$$\{X_a^* = X^*((t-a) \vee 0) : a \geq 0\}$$

for  $x = 0$ , where  $X^*$  is the unique solution starting at 0 such that  $X^*(t) > 0$  for all  $t > 0$ .

If  $\mathcal{C}_P(x)$  denotes the set of all solutions to (4.1) starting at  $x$ , then  $\mathcal{C}_P(x) = \{\delta_{X_x}\}$  for  $x \in (0, 1]$ , where  $\delta_{X_x}$  is the Dirac measure on  $C([0, \infty); \mathbf{R})$  concentrated on  $X_x$ .

If  $x = 0$ , a solution starts at 0 and stays for an arbitrary time, then follows the solution  $X^*$  (suitably translated). So the *departing* time from 0 can be interpreted as a random time whose law can be arbitrary (see fig. 4.1).

**Lemma 4.1.** *The set of solutions starting at  $x = 0$  is given by*

$$\mathcal{C}_P(0) = \left\{ \int \delta_{X_a^*} \mu(da) : \mu \text{ is a probability measure on } [0, \infty] \right\}$$

In conclusion, any selection is completely described by a single random variable on  $[0, \infty)$  (or, equivalently, by a single measure on  $[0, \infty)$ ). Given a probability measure  $\nu$

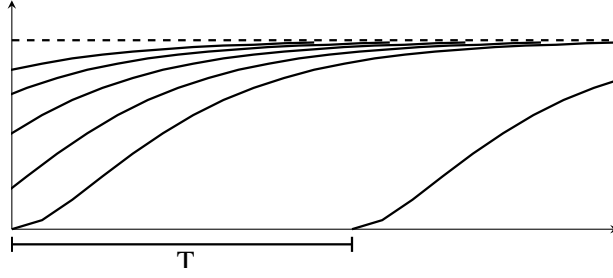


FIGURE 1. Solutions of the Peano example

on  $[0, \infty)$ , define

$$P_x^v = \begin{cases} \delta_{X_x} & x \in (0, 1], \\ \int \delta_{X_a^*} v(da) & x = 0, \end{cases}$$

then  $(P_x^v)_{x \in [0, 1]}$  is a measurable selection, and any selection corresponds to one of them for some  $v$ .

**Proposition 4.2.** *A measurable selection  $(P_x^v)_{x \in [0, 1]}$  is Markov if and only if  $v$  is the distribution of an exponential random variable (including the degenerate cases of infinite or zero rate, where  $v = \delta_0$  or  $v = \delta_\infty$ ).*

*Proof.* It is easy to see that the Markov property holds if  $x \in (0, 1]$  whatever is  $v$ . Indeed,  $P_x^v$ -a. s.,

$$\mathbb{E}^{P_x^v}[f(\xi_{t+s})] = f(X_x(t+s)) = f(X_{X_x(t)}(s)) = \mathbb{E}^{P_x^v}[f(X_{\xi_t}(s))] = \mathbb{E}^{P_x^v}[\mathbb{E}^{P_{\xi_t}^v}[f(\xi'_s)]].$$

We next see which condition we get if  $x = 0$ . On one side,

$$\begin{aligned} \mathbb{E}^{P_0^v}[f(\xi_{t+s})] &= \int f(X_a^*(t+s)) v(da) \\ &= \int_{[0, t)} f(X^*(t+s-a)) v(da) + \int_{[t, +\infty)} f(X^*((t+s-a) \vee 0)) v(da) \\ &= \boxed{1} + \int f(X_b^*(s)) (\theta_t v)(db), \end{aligned}$$

where  $\theta_t : [t, \infty) \rightarrow [0, \infty)$  is defined as  $\theta_t(s) = s - t$ . On the other side,

$$\begin{aligned} \mathbb{E}^{P_0^v}[\mathbb{E}^{P_{\xi_t}^v}[f(\xi'_s)]] &= \int_{[0, t)} \mathbb{E}^{P_{X^*(t-a)}^v}[f(\xi_s)] v(da) + \int_{[t, +\infty)} \mathbb{E}^{P_0^v}[f(\xi_s)] v(da) \\ &= \int_{[0, t)} f(X^*(t+s-a)) v(da) + v([t, \infty)) \int f(X_b^*(s)) v(db) \\ &= \boxed{1} + v([t, \infty)) \int f(X_b^*(s)) v(db). \end{aligned}$$

In conclusion,

$$\int f(X_b^*(s)) (\theta_t v)(db) = v([t, \infty)) \int f(X_b^*(s)) v(db).$$

Moreover, by splitting the integrals in the formula above on  $[0, s)$  and  $[s, \infty]$ ,

$$\begin{aligned} f(0)\phi(s+t) + \int_{[0,s)} f(X^*(s-b))(\theta_t \mathbf{v})(db) &= \\ &= f(0)\phi(s)\phi(t) + \phi(t) \int_{[0,s)} f(X^*(s-b))\mathbf{v}(db), \end{aligned}$$

where  $\phi(r) = \mathbf{v}([r, \infty])$ .

A further simplification can be achieved since  $b \in (0, s] \rightarrow X^*(s-b) \in (0, X^*(s)]$  is invertible (with inverse  $g$  and  $g(0) = s$ ), so that if  $f = F \circ g$ , then we finally get

$$F(s)\phi(s+t) + \int_{[0,s)} F(b)(\theta_t \mathbf{v})(db) = F(s)\phi(s)\phi(t) + \phi(t) \int_{[0,s)} F(b)\mathbf{v}(db).$$

This implies that  $\phi(t)\mathbf{v} = \theta_t \mathbf{v}$  and

$$\phi(s+t) = \phi(s)\phi(t)$$

and  $\mathbf{v}$  is the law of an exponential random variable.  $\square$

For every  $a \in [0, \infty]$  we denote by  $(P_x^a)_{x \in [0,1]}$  the Markov selection of rate  $a$ . We shall call *extremal* all those Markov solutions that can be obtained by the selection procedure outlined in Section 2.

**Proposition 4.3.** *The extremal selections are those corresponding to  $a = 0$  and  $a = \infty$ .*

*Proof.* Given  $\lambda > 0$  and a function  $f$ , a straightforward computation gives

$$\begin{aligned} J_{\lambda,f}(P_0^a) - J_{\lambda,f}(P_0^b) &= \frac{b-a}{(\lambda+a)(\lambda+b)} [\lambda J_{\lambda,f}(P_0^0) - f(0)] \\ &= \frac{\lambda(b-a)}{(\lambda+a)(\lambda+b)} [J_{\lambda,f}(P_0^0) - J_{\lambda,f}(P_0^\infty)], \end{aligned}$$

and with this formula the conclusion is obvious.  $\square$

As it regards invariant measures, we notice that  $(P_x^a)_{x \in [0,1]}$  has a unique invariant measure (which is  $\delta_1$ ) if and only if  $a < \infty$ . Notice that all selections having  $\delta_1$  as their unique invariant measure coincide  $\delta_1$ -almost surely.

If  $a = \infty$ , there are infinitely many invariant measures (the convex hull of  $\delta_0$  and  $\delta_1$ ). As there is no noise in this example, in general we cannot expect the invariant measures to be equivalent (compare with Theorem 2.8).

**4.2. An example of non-uniqueness from Girsanov.** In his paper [14], Girsanov is able to classify the generators of all diffusions which solve the following stochastic differential equation,

$$(4.2) \quad dX_t = \sigma_\alpha(X_t) dW_t,$$

where, for any  $\alpha \in (0, \frac{1}{2})$ ,  $\sigma_\alpha$  is the function

$$\sigma_\alpha(x) = \frac{|x|^\alpha}{1 + |x|^\alpha}.$$

Engelbert & Schmidt [6] give a characterisation for existence and uniqueness of one dimensional SDEs as the one under examination. Their Theorem 2.2 ensures that there is at least one solution for each initial condition, while their Theorem 3.2 implies that the problem has no unique solution.

*Remark 4.4.* The same conclusions hold for a generic function  $\sigma$  such that  $\sigma^{-2}$  is locally integrable and the set of zeroes  $\{x : \sigma^2(x) = 0\}$  is not empty. See also Example 4.1 of Stroock & Yor [24].

In the rest of this section we give a twofold description of Markov solutions to problem (4.2), in terms of the generator and in terms of the process.

4.2.1. *The generators.* Girsanov [14] shows that each of the Markov solutions has its own generator  $\mathcal{L}^\mathbb{G}$  with domain  $D(\mathcal{L}^\mathbb{G})$ . All functions  $C_b^2(\mathbf{R})$  are in  $D(\mathcal{L}^\mathbb{G})$  and for  $x \neq 0$ ,

$$\mathcal{L}^\mathbb{G}u(x) = \sigma_\alpha(x)^2 u''(x).$$

If the solution corresponds to the point 0 to be *absorbing* (i. e., the solutions stays in 0 once it hits it), then

$$\mathcal{L}_\infty^\mathbb{G}u(0) = 0.$$

In the *non-absorbing* case, the generators can be parametrised by  $c \geq 0$ . If  $c > 0$ , the domain  $D(\mathcal{L}_c^\mathbb{G})$  contains all  $C_b^2(\mathbf{R} \setminus \{0\})$  such that the left and right derivatives exist in 0,

$$\mathcal{L}_c^\mathbb{G}u(0) = \frac{1}{c}(u'(0+) - u'(0-)),$$

and  $\mathcal{L}_c^\mathbb{G}u$  is continuous on  $\mathbf{R}$ . In the case  $c = 0$ , which corresponds to a process which spends no time in 0,

$$\mathcal{L}_0^\mathbb{G}u(0) = \lim_{x \rightarrow 0} \sigma_\alpha(x)^2 u''(x).$$

The meaning of the parameter  $c$  will be clarified in the next section, where we shall give an explicit construction of Markov processes solving the problem (see also McKean [19, Section 3.10b]).

4.2.2. *Description of solutions via time-changes and delays.* Following Theorem 2.2 of Engelbert & Schmidt [6], we start by the construction of a process corresponding to  $c = 0$ . Define the (strictly increasing) process

$$S_t^x = \int_0^t \frac{1}{\sigma_\alpha(x + W_s)^2} ds,$$

(the integrability of  $\sigma_\alpha^{-2}$  ensures that  $S_t^x < \infty$  for all  $t \geq 0$ ,  $\mathbb{P}$ -a.s.), and denote by  $T_t^x$  the inverse of  $S^x$ . The process  $(T_t^x)_{t \geq 0}$  is again strictly increasing and  $T_\infty^x = \infty$ . By

Proposition 5.1 of Stroock & Yor [24], the process  $\widehat{X}_t^x = x + W_{T_t^x}$  is a solution to (4.2). Moreover, by Theorem 5.4 of [6], it is the only solution such that

$$(4.3) \quad \mathbb{E} \left[ \int_0^{+\infty} \mathbb{1}_{\{0\}}(X_t) dt \right] = 0.$$

The above condition (4.3) means that  $\widehat{X}^x$  spends no time in 0 and ensures that  $(\widehat{X}_t^x)_{x \in \mathbb{R}}$  is a Markov process.

Any other solution can be obtained by *delaying*  $\widehat{X}$  (Theorem 5.5 of [6]). Indeed, a *time-delay* for  $\widehat{X}$  is any adapted increasing right-continuous process  $(D_t)_{t \geq 0}$  such that

$$D_t = \int_0^t \mathbb{1}_{\{0\}}(\widehat{X}_s) dD_s, \quad t \geq 0, \quad \mathbb{P} - \text{q.c.}.$$

If  $E_t$  is the inverse of  $t \mapsto t + D_t$ , then the process  $Y_t = \widehat{X}_{E_t}$ , adapted to  $\mathcal{F}_t = \mathcal{B}_{E_t}$  is again a solution (Theorem 4.3 of [6]).

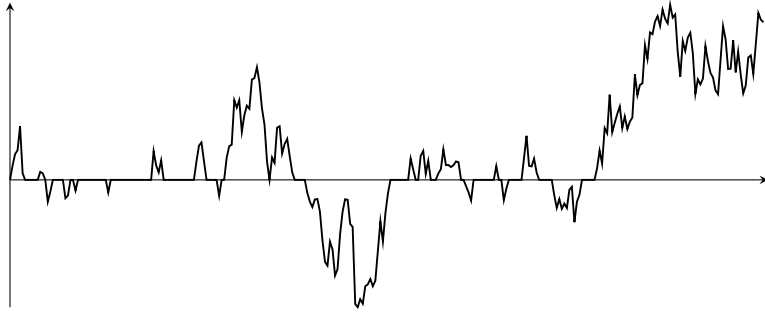


FIGURE 2. The solution is delayed whenever it hits 0

In particular, if  $\tau_0 = \inf\{t \geq 0 : \widehat{X}_t = 0\}$  and  $D_t = 0$  for  $t < \tau_0$ , and  $+\infty$  otherwise, then the delayed process is the process stopped at 0 (which corresponds to the generator  $\mathcal{L}_\infty^c$ ).

Finally, we give an explicit construction (which is taken from Example 6.31 of [6]). Denote by  $L_0$  the local time of  $\widehat{X}$  in 0 and consider  $(S_n, S'_n)$  independent exponential random variables of rate  $\lambda$ , independent from  $\widehat{X}$ . Define  $U_0 = 0$  and

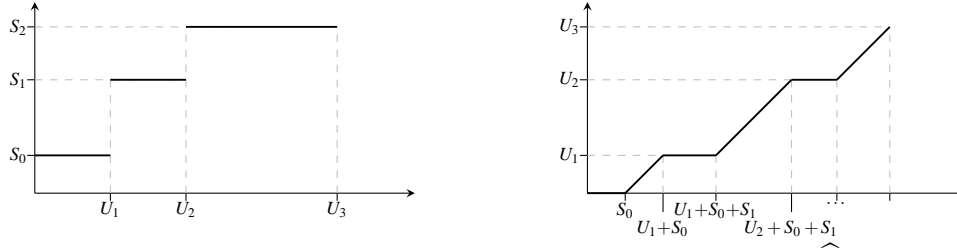
$$U_{n+1} = U_n + \inf\{t \geq 0 : L_0(t + U_n) - L_0(U_n) > S'_n\}.$$

Define finally the time-delay  $(D_t)_{t \geq 0}$  as

$$D_t = S_0 \mathbb{1}_{\{\widehat{X}_0=0\}} + \sum_{k=1}^{\infty} S_k \mathbb{1}_{[U_k, +\infty)}(t)$$

and denote by  $(E_t)_{t \geq 0}$  the inverse of  $t + D_t$  (see Figure 3). The process  $Y_t^x = \widehat{X}_{E_t}$  is a Markov process.

In few words,  $D_t$  jumps (and so  $Y_t^x$  stops at zero for an amount of time corresponding to the size  $S$  of the jump) every time the local time  $L_0$  accumulates enough *mass* (in terms of random variables  $S'$ ).

FIGURE 3. A picture of  $D_t$  (left) and of  $E_t$  (right) when  $\widehat{X}_0 = 0$ .

*Remark 4.5.* A similar example can be considered in dimension 2 (or more), but the behaviour of solutions is slightly different, see Example 4.12 of Stroock & Yor [24].

4.2.3. *Analysis of invariant measures in a damped version.* Similar conclusions can be drawn for the damped problem,

$$d\widetilde{X}_t = -\widetilde{X}_t dt + \sigma_\alpha(\widetilde{X}_t) dW_t.$$

Indeed, we can use the method of *removal of drift* (see for example Proposition 5.13 in Karatzas & Shreve [17]), and reduce the problem to an equation of the same type of (4.2), with a different diffusion coefficient, which anyway has exactly the same regularity properties as  $\sigma_\alpha$ . This is possible since  $\frac{b(x)}{\sigma_\alpha(x)^2}$  is a bounded function, where  $b$  is the drift function  $b(x) = -x$ .

It is easy to verify that each Markov solution has a unique invariant measure. Each of these measures, with the exception of the one corresponding to the Markov process which spends no time in 0, has an atom in 0. In particular, there are invariant measures that are not mutually equivalent.

4.3. **A strong Feller example by Stroock and Yor.** Following Example 4.5 of Stroock & Yor [24], consider the following diffusion operator,

$$\mathcal{L}_{\text{SY}} = \frac{1}{2} \mathbb{1}_G(x) \frac{\partial^2}{\partial x^2} + \mathbb{1}_{\{0\}}(x) \frac{\partial}{\partial x},$$

where  $G = \mathbf{R} \setminus \{0\}$ , and denote by  $\mathcal{C}_{\text{SY}}(x)$ , for every  $x \in \mathbf{R}$ , the set of all probability measures on  $C([0, \infty); \mathbf{R})$  solutions to the martingale problem associated to  $\mathcal{L}_{\text{SY}}$ .

For every  $x \in \mathbf{R}$ , denote by  $W_x$  the law of  $x + B_t$ , where  $(B_t)_{t \geq 0}$  is a one-dimensional standard Brownian motion (hence  $W_x$  is the Wiener measure at  $x$ ) and it is clear that  $W_x \in \mathcal{C}_{\text{SY}}(x)$  for all  $x$ . In particular,  $(W_x)_{x \in \mathbf{R}}$  is a Markov solution to the problem which is *strong Feller*.

The problem is not well posed and it is possible to see that there is another strong Feller Markov solution, corresponding to the reflected Brownian motion. We give a few hints, all details can be found in Stroock & Yor [24, Example 4.5].

First, by Lemma 4.6 of [24], a probability measure  $P \in \mathcal{C}_{\text{SY}}(x)$  if and only if

1.  $P[\xi_0 = x] = 1$ ,



2. for every  $\phi \in C^{1,2}([0, \infty) \times \mathbf{R})$  such that  $\partial_t \phi(t, 0) + \partial_x \phi(t, 0) \geq 0$ ,

$$\phi(t, \xi_t) - \int_0^t \mathbb{1}_G(\xi_s) [\partial_t \phi(s, \xi_s) + \frac{1}{2} \partial_x^2 \phi(s, \xi_s)] ds$$

is a  $P$ -submartingale with respect to the natural filtration  $(\mathcal{B}_t)_{t \geq 0}$ .

By Theorems 3.1 and 5.5 of Stroock & Varadhan [22], for every  $x \geq 0$  there exists a unique probability measure  $Q_x$  such that  $Q_x \in \mathcal{C}_{SV}(x)$  and

$$(4.4) \quad Q_x[\xi_t \geq 0 \text{ for all } t \geq 0] = 1.$$

Define  $\tau_0$  as the hitting time of 0. If  $x < 0$ , define  $Q_x$  as the probability measure equal to  $W_x$  up to time  $\tau_0$ , and then equal to  $Q_0$  suitably translated to time  $\tau_0$  afterwards. Property (4.4) ensures that the solution  $(Q_x)_{x \in \mathbf{R}}$  is Markov. Moreover, since  $Q_x = W_x$  up to time  $\tau_0$ , it follows that

$$\mathbb{E}^{Q_x}[\phi(\xi_t)] = \mathbb{E}^{W_x}[\phi(\xi_t) \mathbb{1}_{\{\tau_0 > t\}}] + \mathbb{E}^{W_x}[\tilde{\mathcal{P}}_{t-\tau_0} \phi(0) \mathbb{1}_{\{\tau_0 \leq t\}}],$$

for every  $\phi \in C_b(\mathbf{R})$ , where  $(\tilde{\mathcal{P}}_t)_{t \geq 0}$  is the transition semigroup associated to  $(Q_x)_{x \in \mathbf{R}}$ . Hence  $(Q_x)_{x \in \mathbf{R}}$  is also strong Feller.

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